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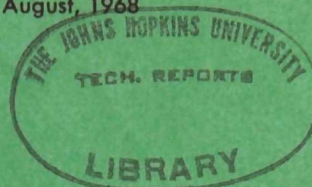
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THE JOHNS HOPKINS UNIVERSITY

Department of Mechanics

Baltimore, Maryland

August, 1968



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A SIMILARITY SOLUTION FOR THE GULF STREAM

By

Gerald S. Janowitz

Technical Report No. 30 (WB-ESSA Series)

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THE U. S. DEPARTMENT OF COMMERCE
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Abstract

A simple model of the Gulf Stream is considered which allows us to study in detail the inertial, viscous, and rotational effects in a narrow two-dimensional current, flowing northwards along the western edge of a rectangular ocean basin, under the influence of a uniform inflow from the interior of the basin. It is found that in the region of maximum northwards velocity all of the forces are significant. A method for estimating the eddy viscosity is also discussed.

A SIMILARITY SOLUTION FOR THE GULF STREAM

1. Introduction

Previous detailed studies of the Gulf Stream have either considered linear frictional models, e.g., Munk (1950), or non-linear inviscid models, e.g., Charney (1955) and Morgan (1956). Moore (1963) constructed a model of the general circulation in an ocean basin which includes viscous effects and inertial effects through an Oseen linearization. His paper demonstrates the significance of the Reynolds number $\frac{U}{\nu} \left(\frac{U}{\beta} \right)^{1/2}$, where U is a velocity typical of the overall circulation. His theory does not attempt to consider details of the flow.

This paper considers a two-dimensional model of the Gulf Stream with constant density and eddy viscosity, in which the effect of the interior is to supply a uniform inflow to the current. We shall be able to study in detail the relative significance of the various forces which are acting and, thus, hope to provide information and direction for more realistic, i.e., stratified, models.

The ocean basin is idealized as rectangular and of uniform depth, H , the fluid is of uniform density and uniform eddy viscosity, and the flow is parallel to the surface. The origin of a rectangular cartesian coordinate system is placed in the southwestern corner of the basin with the y -axis running northwards along a rigid wall and the x -axis eastwards. A uniform flow of speed U approaches the wall from the east. The northwards velocity along the x -axis is assumed to vanish (Fig. 1). Making the beta-plane approximation, the governing equations

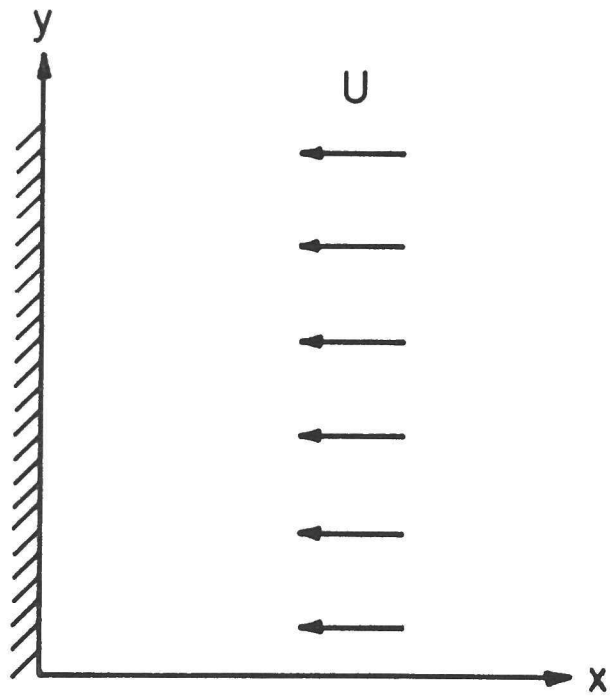


Figure 1. Geometry of the problem

become

$$u u_x + v u_y - (f_0 + \beta y) v = -\frac{1}{\rho} P_x + \nu \nabla^2 u \quad (1a)$$

$$u v_x + v v_y + (f_0 + \beta y) u = -\frac{1}{\rho} P_y + \nu \nabla^2 v \quad (1b)$$

$$u_x + v_y = 0 \quad (1c)$$

The boundary conditions become

$$\begin{aligned} u(0, y) = v(0, y) &= 0 \\ v(x, 0) &= 0, \end{aligned} \quad (1d)$$

and as $x/L \rightarrow \infty$

$$u \rightarrow -U, \quad v \rightarrow 0.$$

where L is the width of the stream. We now assume that within the stream

$$\frac{\partial}{\partial x} \gg \frac{\partial}{\partial y}$$

We define a streamfunction by

$$u = -\psi_y, \quad v = \psi_x. \quad (2)$$

Differentiating (1b) and (1a) with respect to x and y respectively and subtracting the results leads to the following equation for the streamfunction

$$\frac{d}{dt} (\psi_{xx} + \beta y) = \nu \psi_{xxxx}, \quad (3)$$

or

$$-\psi_y \psi_{xxx} + \psi_x \psi_{xxy} + \beta \psi_x = \nu \psi_{xxxx}. \quad (4)$$

We seek the most general similarity solution of this equation which is of the form

$$\psi = g(y) f(\eta) \quad (5a)$$

where

$$\eta = \alpha x k(y) \quad (5b)$$

and α is a constant to be chosen for convenience. Substituting Eqs. (5) into (4) leads to

$$\psi = \left(\frac{\nu^2 \beta}{c_2^4 c_1^2} \right)^{1/3} (y + c_3) f(\eta) \quad (6a)$$

where

$$\eta = \alpha \left(\frac{\beta c_2^2}{\nu c_1^2} \right)^{1/3} x \quad (6b)$$

and to the following equation for $f(\eta)$

$$-f f'''' + f' f'' + \frac{c_1^2}{\alpha^2} f = \alpha c_2^2 f'''' \quad (7)$$

where C_1 , C_2 and C_3 as well as α are arbitrary constants. To simplify Eq. (7) we choose

$$\alpha = C_1$$

Applying conditions (1d), we find

$$f(0) = 0, \quad (8a)$$

$$f'(0) = 0, \quad (8b)$$

$$f'(\infty) = 0, \quad (8c)$$

$$f(\infty) = \frac{U}{(\nu^2 \beta)^{1/3}} (c_1 c_2^2)^{2/3}.$$

We choose

$$(c_1 c_2^2)^{-1} = \frac{U^{3/2}}{\nu \beta^{1/2}}$$

so

$$f(\infty) = 1.0 \quad (8d)$$

Equation (6b) simplifies to

$$\eta = \left(\frac{\beta}{U}\right)^{1/2} x \quad (9)$$

We make the following definitions:

$$L_x \equiv \left(\frac{U}{\beta}\right)^{1/2}, \quad R \equiv \frac{U L_x}{\nu} = \frac{U^{3/2}}{\nu \beta^{1/2}} \quad (10)$$

Equations (6a), (2), and (7) become

$$\begin{aligned} \psi &= U y f\left(\frac{x}{L_x}\right) = U y f(\eta), \quad u = -U f(\eta) \\ v &= \frac{U y}{L_x} f'(\eta), \end{aligned} \quad (11)$$

and

$$-f f''' + f' f'' + f' = \frac{1}{R} f''''$$

This governing equation for $f(\eta)$ may be integrated from zero to η

and we obtain

$$f'^2 - f f'' + f = \frac{1}{R} f''' - \frac{1}{R} f'''(0)$$

Letting $\eta \rightarrow \infty$, and applying the boundary conditions, we find

$$f'''(0) = -R$$

Finally the governing equation for $f(\eta)$ becomes

$$f'^2 - f f'' + f = \frac{1}{R} f''' + 1 \quad (12)$$

Equation (12) has been discussed by Moore (1963), under a different set of boundary conditions, where the flow near a latitude of vanishing wind stress curl was under consideration. He discussed the asymptotic solution of this equation and did not attempt to integrate it.

The pressure field can be computed from Eqs. (1a) and (1b) and we find

$$\frac{p(x, y) - p_{\pm}(\infty, y)}{\rho U^2} = \left[\frac{1}{2}(1-f^2) - \frac{f'}{R} \right] - (1-f) \frac{f_0 + \beta y}{U} y, \quad (13)$$

and

$$p_{\pm}(\infty, y) = p_{\pm}(\infty, 0) + \rho U \left(f_0 + \frac{\beta y}{2} \right) y, \quad (14)$$

where $p_{\pm}(\infty, y)$ is the pressure at the outer edge of the current.

2. Solutions for large η

We now wish to study the asymptotic nature of the solution. We define $g(\eta)$ by

$$f(\eta) = 1 + g(\eta) \quad (15)$$

Assuming that $f(\eta)$ is a solution to Eq. (12) under conditions (8), for sufficiently large η , $g(\eta)$ and its derivatives must become small. For these values of η , $g(\eta)$ must satisfy

$$\frac{1}{R} g''' + g'' - g = 0 \quad (16)$$

Seeking solutions of $g(\eta)$ which remain small and are of the form $e^{\alpha \eta}$, requires that α have a negative real part and satisfy

$$\frac{\alpha^3}{R} + \alpha^2 - 1 = 0.$$

We may readily show the following.

(a) If $R < \frac{\sqrt{27}}{2} \approx 2.60$, the decaying root is complex, and we may say that a countercurrent exists.

(b) If $R > \frac{\sqrt{27}}{2}$, the decaying root is real and no countercurrent exists. Hence, it may be possible that the real Gulf Stream exhibits a countercurrent at some times and none at others, depending on the value of R .

3. A Linear Solution

In this Section we rederive Munk's solution to obtain numerical values to compare with the results of the integration of (12). We neglect inertial terms on the left-hand side of (1a) and (1b) and make the boundary-layer approximation. The governing equation becomes

$$\beta \psi_x = \nu \psi_{xxxx}$$

We seek a solution of the form

$$\psi = U_y F(\eta), \quad \eta = \left(\frac{\beta}{\nu}\right)^{1/2} x = \frac{x}{L_x},$$

which satisfies the conditions

$$\psi(0, y) = \psi_x(0, y) = 0, \quad \psi_x(\infty, y) = 0, \quad \psi_y(\infty, y) = U.$$

The solution is

$$F(\eta) = 1 - \frac{2}{\sqrt{3}} e^{-\frac{R^{1/3} \eta}{2}} \cos\left(\frac{\sqrt{3}}{2} R^{1/3} \eta - \pi/6\right) \quad (18)$$

This is Munk's (1950) solution. This solution cannot directly be thought of as the low Reynolds number solution of (12), since as $R \rightarrow 0$, $F(\eta) \rightarrow 0$. However, if we change the independent variable of (12) to η_M , where

$$\eta_M \equiv R^{1/3} \eta = \left(\frac{\beta}{\nu}\right)^{1/3} x,$$

and denote $\frac{d}{d\eta_M}$ by $(\dot{})$, Eq. (12) becomes

$$\ddot{f} + 1 = f + R^{2/3} (\dot{f}^2 - f \ddot{f}). \quad (19)$$

Then clearly

$$F(\eta_M) = 1 - \frac{2}{\sqrt{3}} e^{-\frac{\eta_M}{2}} \cos\left(\frac{\sqrt{3}}{2} \eta_M - \frac{\pi}{6}\right)$$

is the solution of (19) as $R \rightarrow 0$.

Equation (18) provides numerical information which we can use as a comparison with the solutions of (12).

First, we define the strength of the countercurrents, S , as the value of $f(\eta) - 1$ at the first zero of $f'(\eta)$. For Munk's solution

$$S = F\left(\frac{2\pi}{\sqrt{3} R^{1/3}}\right) - 1 = 0.163 \quad (20)$$

The maximum value of $f'(\eta)$, which is proportional to the maximum northwards velocity is

$$F'_{\max}(\eta) = \frac{v_{\max}(\eta) L_x}{\psi(\infty, y)} = 0.547 R^{1/3} \quad (21)$$

This maximum of $F'(\eta)$ occurs at

$$\eta = 1.21 R^{-1/3} \quad (22)$$

The relative vorticity at the wall is proportional to $f''(0)$. We find

$$F''(0) = \frac{\frac{\partial v}{\partial x}(0, y)}{\beta y} = R^{2/3} \quad (23)$$

The information contained in Eqs. (20)-(23) will be compared to the numerical results.

4. The Numerical Integration

Equation (12) and the conditions

$$f(0) = f'(0) = 0, \quad f(\infty) = 1.0$$

form a two-point boundary value problem. We chose to guess at $f''(0)$

and to integrate (12) outwards to a large value of η (≈ 5.0). The solution was assumed to be obtained when $f(5.0) \approx 1.0$ and $f'(5.0)$,

$f''(5.0) \ll 1$. . For small R , Eq. (19) was integrated out to

$\eta_M \approx 5.0$. Actually, it was found that if the integration of (12)

were continued, for sufficiently large η , the solution blew up. However,

numerical integrations of (12) corresponding to bracketing values of

$f''(0)$ blew up much more rapidly and in different manners (Fig. 2),

and it was not difficult to find the solution.

5. The Values of R chosen for the Integration.

We chose to integrate (12) for $R = 0.1, 1.0, 5.0, 10.0$.

This range enabled us to consider cases with countercurrents ($R = 0.1, 1.0$) as well as more realistic low eddy viscosity cases. We now

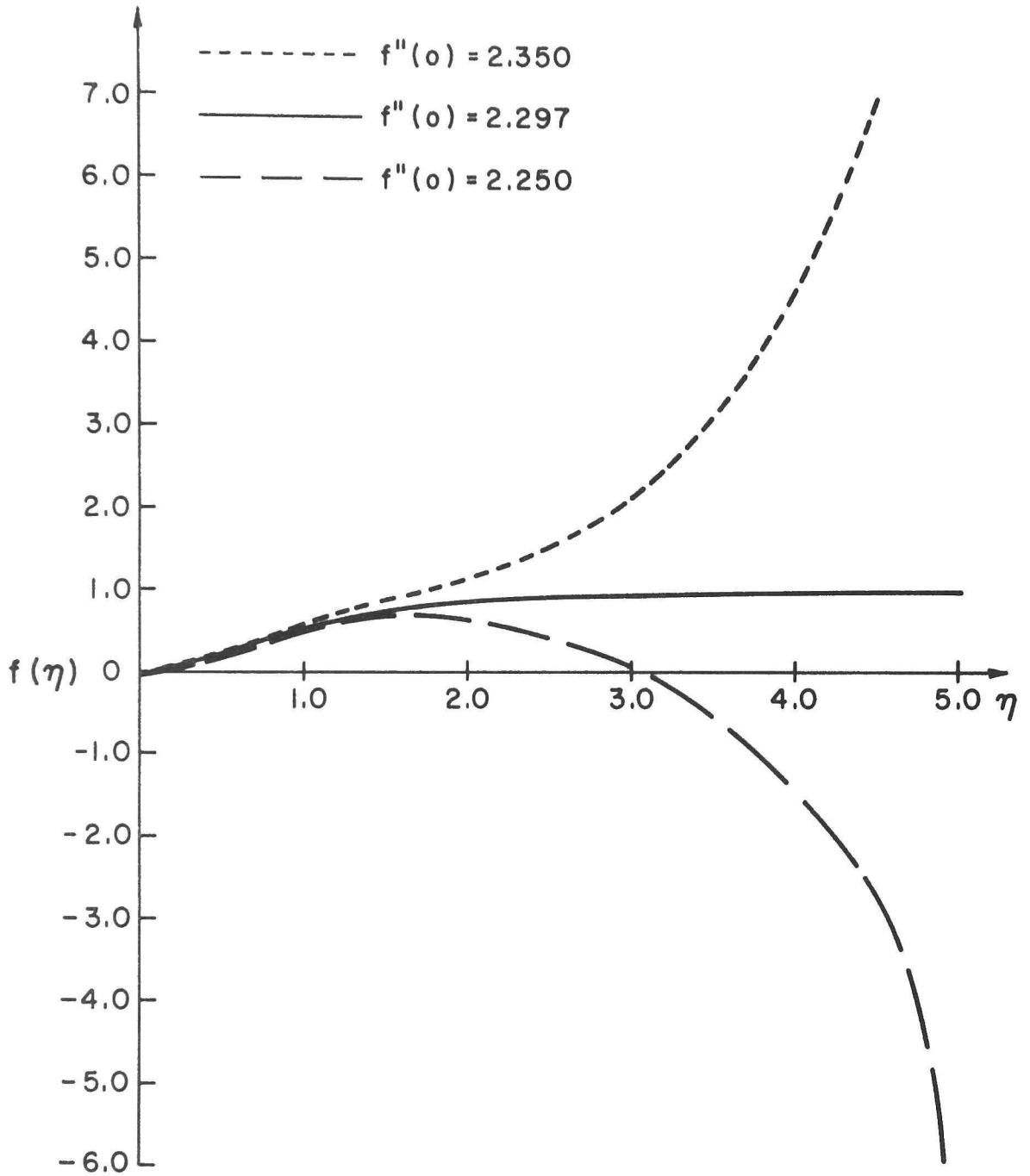


Figure 2. Plot of $f(\eta)$ vs. η for $R = 5.0$ and three values of $f''(0)$.

consider the ranges of H and V that this range of R encompasses.

We first estimate V in terms of H . At 35° N. latitude, the Gulf Stream has been flowing for about 20° of latitude ($\sim 20 \times 10^7 \text{ cm.}$) is of depth H , and carries approximately $74 \times 10^{12} \text{ cm}^3$ of fluid. Thus,

$$V \times (20 \times 10^7 \text{ cm}) \times H = 74 \times 10^{12} \text{ cm}^3/\text{sec},$$

or

$$V = \frac{3.7}{H} \text{ cm/sec}$$

if H is expressed in kilometers. Taking $\beta = 1.9 \times 10^{-13} / \text{cm. sec.}$ we find

$$R = \frac{1.6 \times 10^7 \text{ cm}^2/\text{sec}}{H^{3/2} V}$$

and

$$L_x = \frac{44.1}{H^{1/2}} \text{ Km}$$

Taking two reasonable values for H , Fig. 3 gives the range of V and L_x for the range of R considered.

H (Km.)	V (cm. ² /sec.)		L_x (Km.)
	$R=0.1$	$R=10.0$	
0.50	4.6×10^8	4.6×10^6	62.3
1.00	1.6×10^8	1.6×10^6	44.1

Figure 3. The range of H and V .

Thus, the range of R considered allows a wide variation in V and H to be considered.

6. The Results

After a number of iterations, solutions to Eq. (12) under conditions (8) were obtained. As expected, for $R=0.1$ and $R=1.0$ countercurrents exist. The strengths of these countercurrents, S , as defined in Section 3 are given in Fig. 4.

	MUNK'S THEORY	$R=0.1$	$R=1.0$
$S=$	0.163	0.111	0.018

Figure 4. The strengths of the countercurrents. $S = f(\bar{\eta}) - 1$, $f'(\bar{\eta}) = 0$. For higher R , $f(\bar{\eta})$ approaches 1.0 monotonically, and no countercurrents exist. The velocity profiles are plotted in Figs. 6-9.

In Fig. 5 we show the maximum northwards velocity and the result according to the linear theory (Eq. (21)). We see that the linear theory overestimates this velocity.

$R=$	0.1	1.0	5.0	10.0
EXACT RESULT	0.239	0.443	0.608	0.674
LINEAR THEORY	0.254	0.547	0.936	1.175

Figure 5. The maximum value of $f'(\eta)$ ($= \frac{v(\eta) L_x}{\psi(\infty, y)}$).

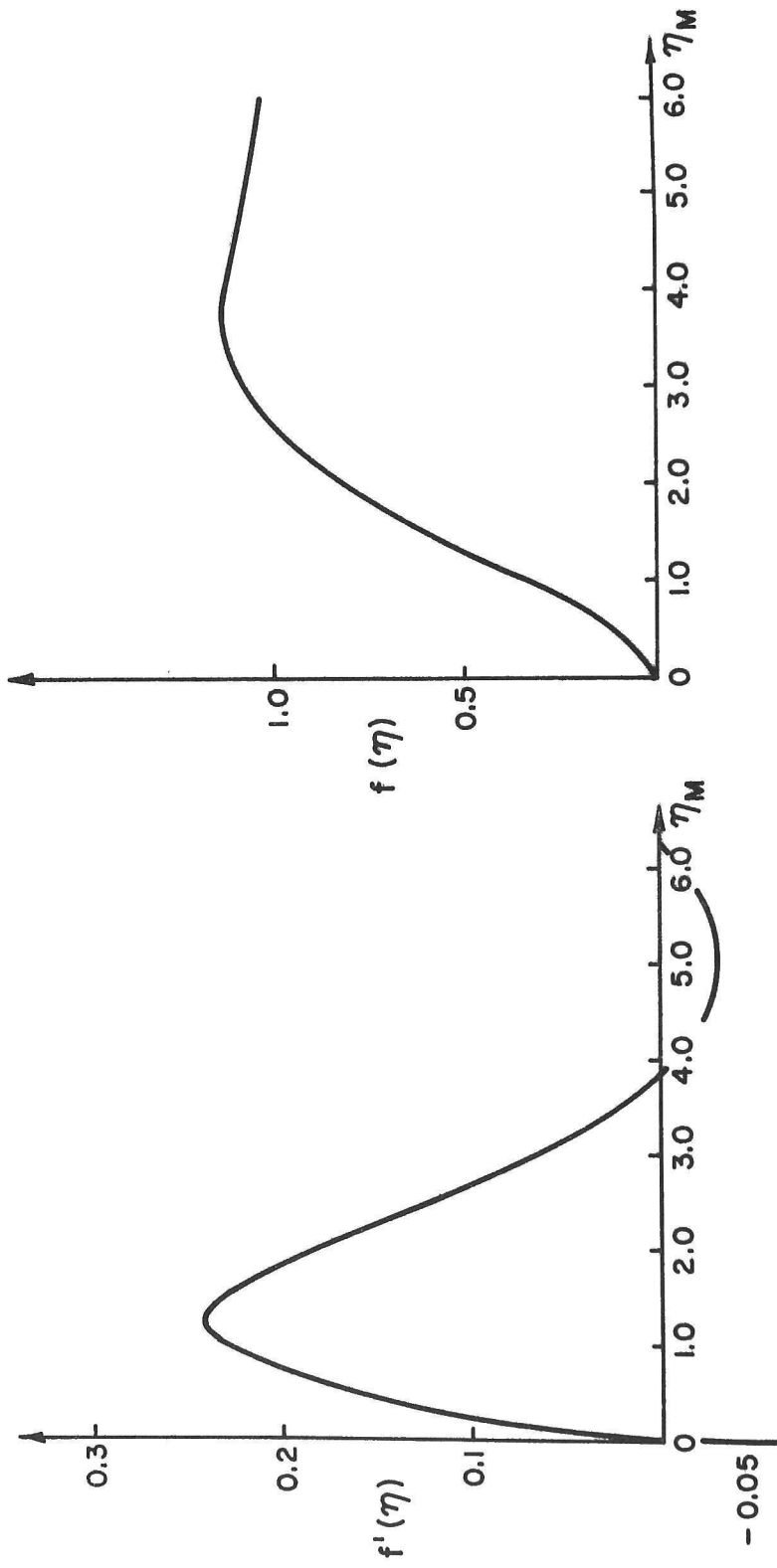


Figure 6. Plot of $f(\eta)$ and $f'(\eta)$ for $R = 0.10$. $f'(\eta) = v L_x / U_y$, $f(\eta) = -\frac{u}{U}$.

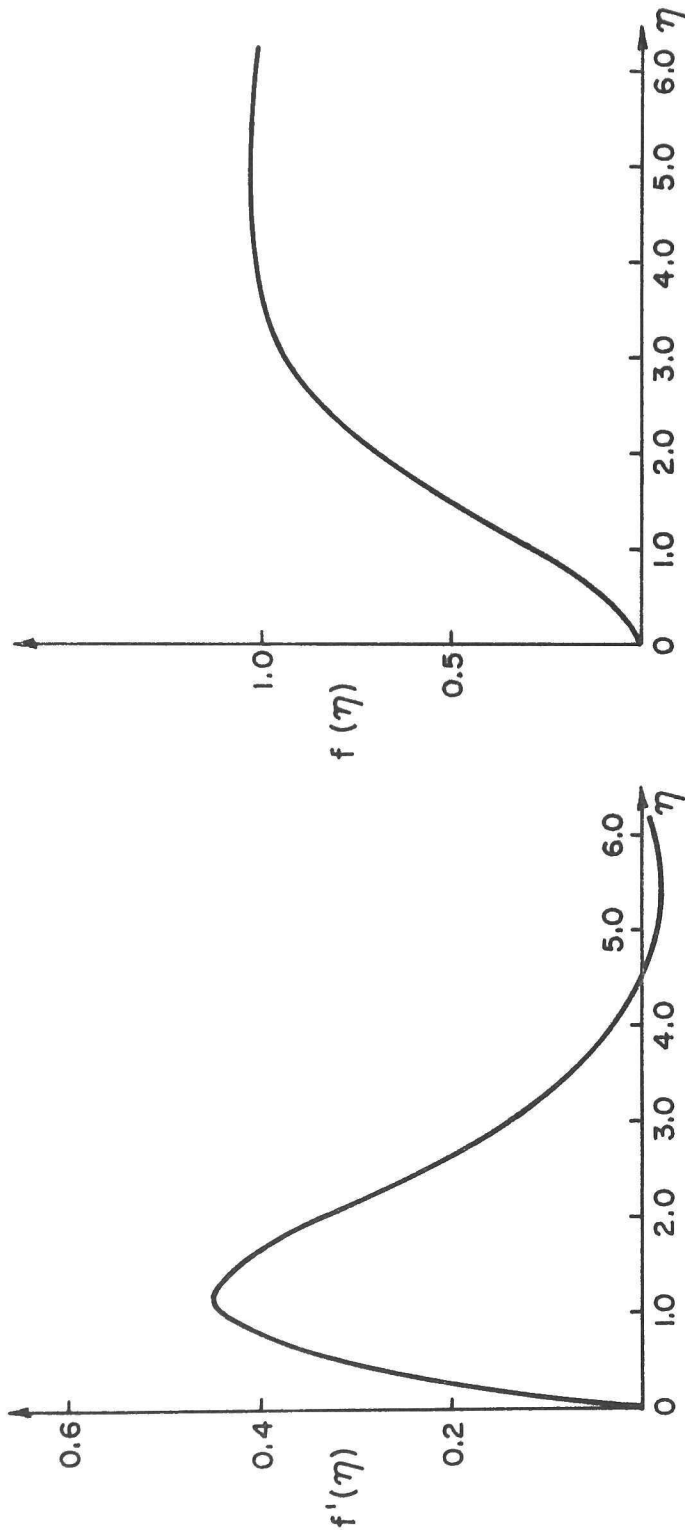


Figure 7. Plot of $f'(\eta)$ and $f(\eta)$ for $R = 1.0$. $f'(\eta) = v_L x / U y$, $f(\eta) = -\frac{y}{U}$.

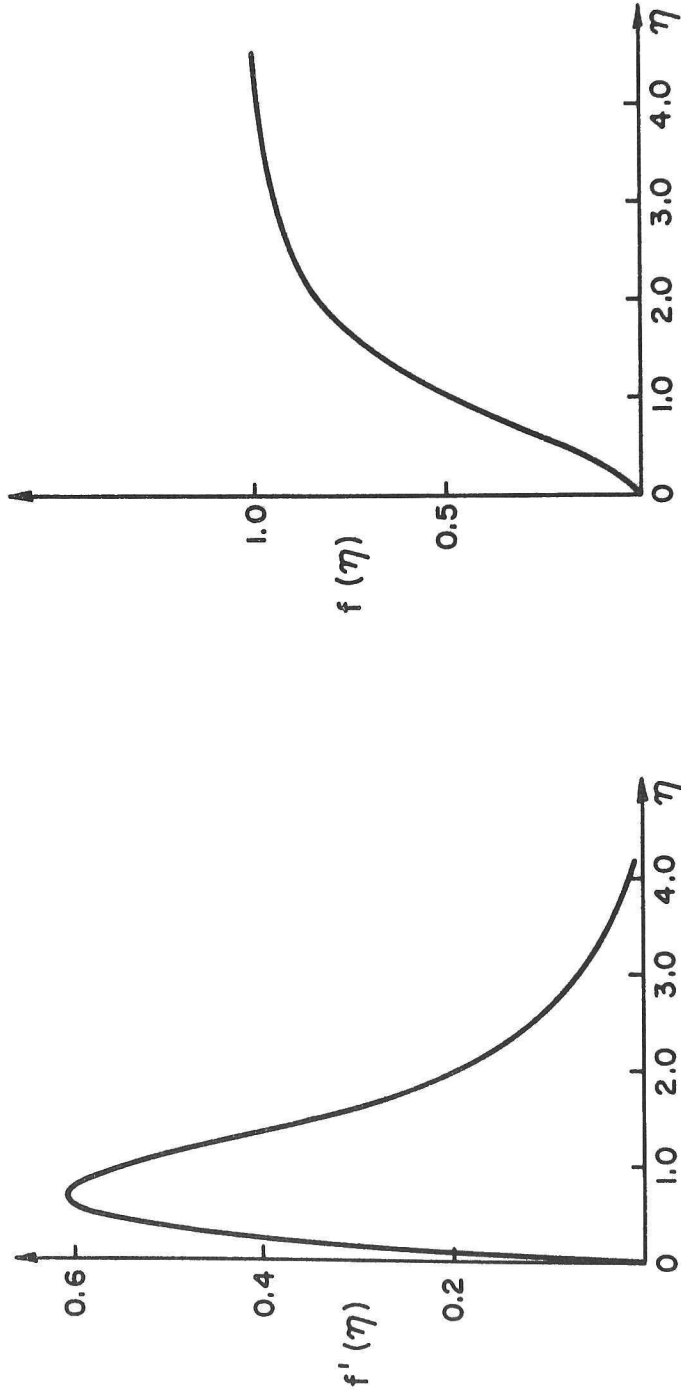


Figure 8. Plot of $f(\eta)$ and $f'(\eta)$ for $R = 5.0$. $f'(\eta) = v L_x / U y$, $f(\eta) = -\frac{u}{U}$

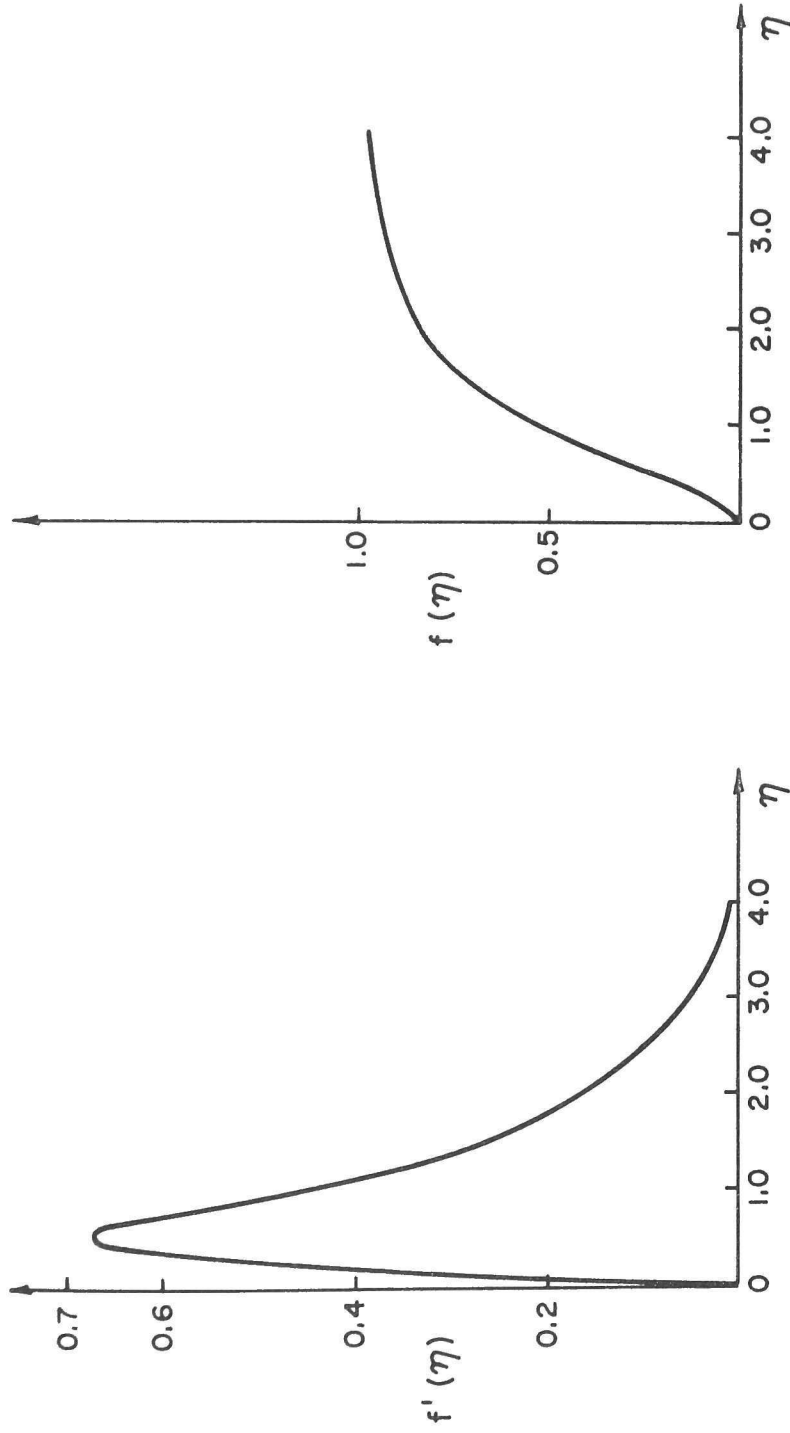


Figure 9. Plot of $f(\eta)$ and $f'(\eta)$ for $R = 10.0$ $f'(\eta) = v_{Lx}/Uy$, $f(\eta) = -\frac{u}{U}$.

One measure of the width of the current is given by the value of η where U is a maximum. This is given in Fig. 10 along with the result from the linear theory (Eq. (22)). The linear result again overestimates the value obtained numerically.

$R =$	0.1	1.0	5.0	10.0
EXACT RESULT	2.59	1.13	0.62	0.46
LINEAR THEORY	2.61	1.21	0.71	0.56

Figure 10. The value of η where U is a maximum.

Another result of interest is $f''(0)$ since this is proportional to the relative vorticity at the wall as well as the viscous force exerted by the fluid on the wall. It is also the value which we use to begin the numerical integration. The actual value, as well as the linear result (Eq. 23), is shown in Fig. 11.

$R =$	0.1	1.0	5.0	10.0
EXACT RESULT	0.2085	0.8887	2.297	3.397
LINEAR THEORY	0.2155	1.000	2.920	4.640

Figure 11. The values of $f''(0) (= \frac{\partial U}{\partial x}(0, y) / \beta y)$.

The linear result overestimates the actual value but gives a reasonable starting point for the integration.

We now define the width of the current, η_w , by requiring

$$f(\eta_w) = 0.9.$$

The numerical integration gives the following (Fig. 12). It is

$R =$	0.1	1.0	5.0	10.0
$\eta_w =$	4.90	2.82	2.49	2.47

Figure 12. The width of the stream, η_w . $f(\eta_w) = 0.9$.

interesting to note that increasing R from 5 to 10 makes less than a 1% change in η_w . A physical measurement of the width of the stream, combined with the assumption that $\eta_w \cong 2.50$ gives a value for L_x , hence for U . A physical measurement of the point of maximum northwards velocity together with the deduced value of L_x gives the value of R for the flow and, hence, an estimate of \mathcal{V} .

We now note that Eq. (12) is identical with the momentum equation in the y -direction, which we can readily see if we substitute (11) into (1b). In Eq. (12) we can identify $1-f$ with the sum of the Coriolis and pressure forces, $f'^2 - ff''$ with the inertial effect, and $\frac{f'''}{R}$ with the viscous force. We designate these three effects as β , I and V respectively. Equation (12) can be written symbolically

$$I = \beta + V$$

To determine the relative importance of these three effects, we plot the numerical results for these terms versus η in Fig. 13, for the range

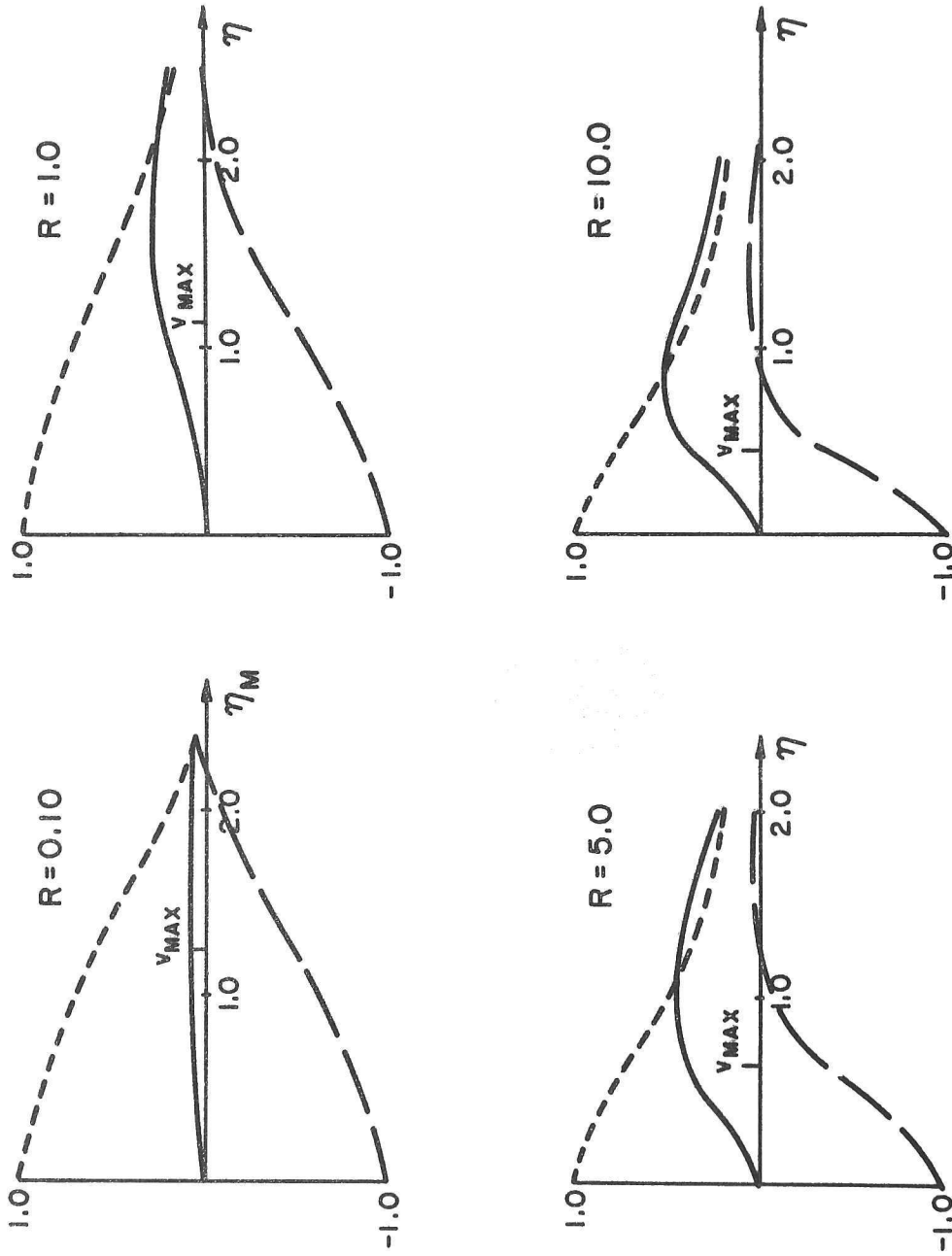


Figure 13. The magnitudes of the various terms in the governing equation. β ---; I —; V —.

of R considered. Near the wall the balance is essentially between β and V . For $R = 1.0, 5.0$, and 10.0 , all three effects are significant in the range of η across the region of maximum northwards velocity. Thus, the hope of matching a linear viscous boundary layer near the wall to an inviscid inertial layer at the outer edge of the stream does not seem to be a reasonable method of studying the details of the motion.

The Oseen approximation has been used by Moore to consider the overall nature of the flow in an ocean basin. To determine how close to the wall this approximation remains valid, in Fig. 14 we plot $\frac{-Uv_x}{uv_x + v^2v_y}$ versus η . We see that for $R = 5.0$ and 10.0 , the Oseen approximation, as measured by this ratio, is valid for $\eta \geq 1.0$.

To summarize, the similarity solution indicates that any realistic model of the Gulf Stream should contain both inertial and viscous effects. The relative insensitivity of the width of the stream to the value of R , at least if $R \geq 5.0$, may lead to estimates of the eddy viscosity based on measurements of the width of the stream and the point of maximum velocity.

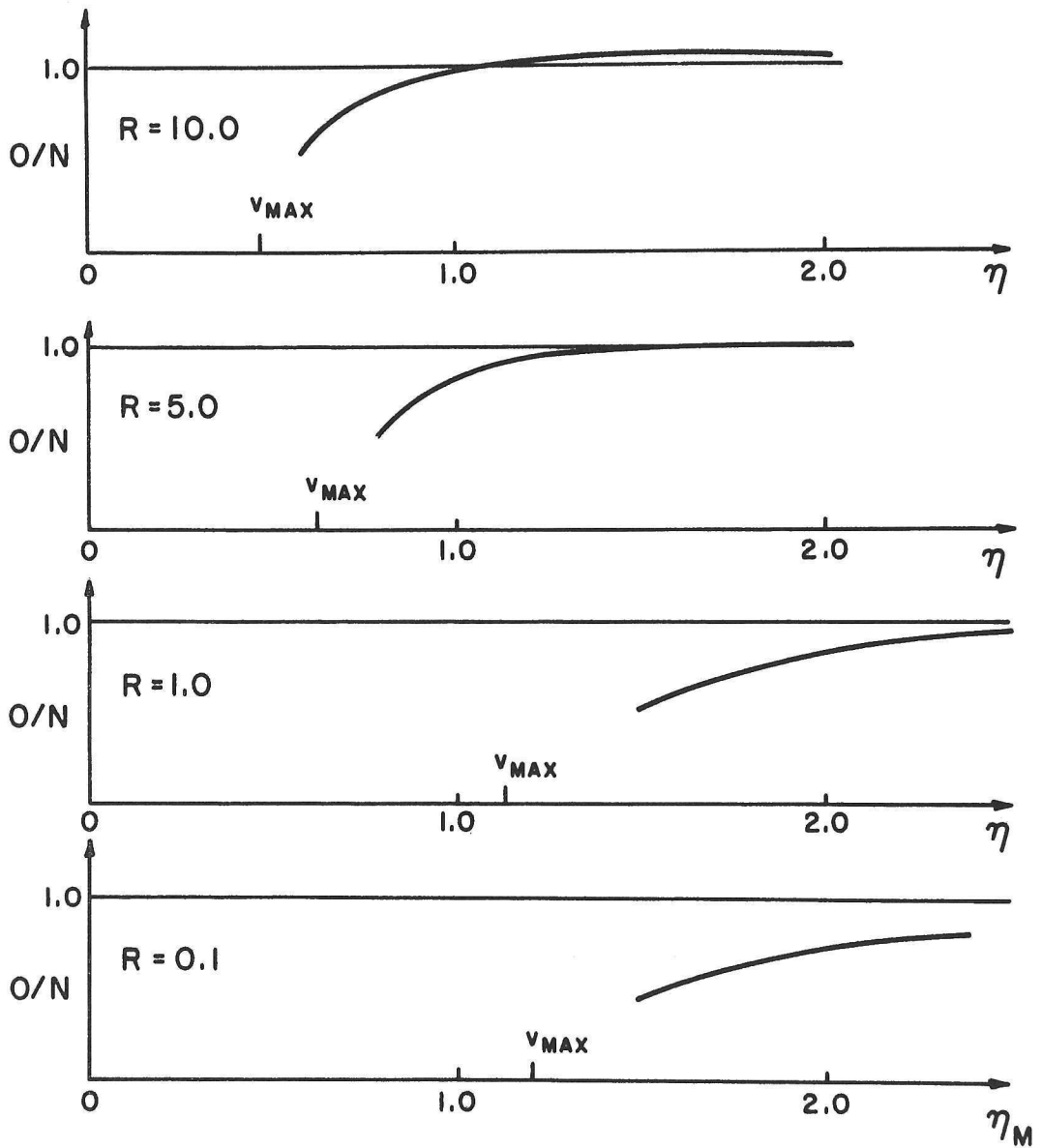


Figure 14. The validity of the Oseen approximation as measured by the ratio O/N .

$$O/N = -Uv_x / (uv_x + vv_y)$$

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